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On well quasiordering of finite languages

Martin Klazar*

Department of Applied Mathematics of Charles University, Malostranské náměstí 25, 11800 Praha 1, Czech Republic

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Abstract

We investigate here the quasiordering $\preccurlyeq$ of finite sets of finite strings over an infinite set of symbols S. We set $\mathscr{K} \preccurlyeq \mathscr{L}$ iff it is possible to rename symbols occurring in the strings of $\mathscr{L}$ so that any string of $\mathscr{K}$ is a subsequence of a string of the renamed $\mathscr{L}$. We prove that $\preccurlyeq$ is a wqo which answers the question raised by Gustedt (1992). We prove also a stronger version with injective correspondence between strings.

1. Introduction

Strings are finite sequences over S where S is an infinite countable set of *symbols*. *Languages* are finite sets of strings, *babels* are sets of languages. If $A \subseteq S$ then A^* stands for the set of all strings over A. By S^{**} we denote the babel consisting of all languages. We define, for a string $u = a_0 a_1 \ldots a_m$, $S(u) = \bigcup_{i=0}^{m} \{a_i\}$. Similarly, $S(\mathscr{L}) = \bigcup_{u \in \mathscr{L}} S(u)$ for any language $\mathscr{L}$.

The notation $u \subset v$ means, for any two sequences $u = a_0 a_1 \ldots a_m$ and $v = b_0 b_1 \ldots b_n$, that u is a subsequence of v: $a_0 = b_{j_0}$, $a_1 = b_{j_1}, \ldots, a_m = b_{j_m}$ for some m indices $0 \leqslant j_0 < j_1 < \cdots < j_m \leqslant n$. We define, for two languages $\mathscr{L}$ and $\mathscr{K}$, that $\mathscr{L} \leqslant \mathscr{K}$ (via f) iff $u \subset f(u)$ for any $u \in \mathscr{L}$ for some mapping $f : \mathscr{L} \to \mathscr{K}$. A mapping $\varphi : S \to S$ transforms a language $\mathscr{L}$ to the language $\varphi(\mathscr{L}) = \{\varphi(u) \mid u \in \mathscr{L}\}$ where $\varphi(u) = \varphi(a_0 a_1 \ldots a_m) = \varphi(a_0)\varphi(a_1) \ldots \varphi(a_m)$. We shall investigate the following quasiordering.

Definition 1.1. $\mathscr{L} \preccurlyeq \mathscr{K}$, $\mathscr{L}$ and $\mathscr{K}$ are languages, iff $\mathscr{L} \leqslant \varphi(\mathscr{K})$ for some $\varphi : S \to S$.

The above quasiordering was introduced in [5] to generalize *chain minor* ordering of finite posets. We say, in accordance with [5] and with [2] that P is a chain minor of

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* E-mail: klazar@kam.ms.mff.cuni.cz.

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Q (P and Q are finite posets) iff there is a mapping $\rho: Q \to P$ such that any chain in P is isomorphic via ρ to a chain in Q (thus ρ must be onto). Chain minor ordering was introduced in connection with scheduling stochastic project networks [5]. Clearly, P is a chain minor of Q iff $\mathscr{L}(P) \preccurlyeq \mathscr{L}(Q)$ where $\mathscr{L}(P)$ and $\mathscr{L}(Q)$ are languages consisting of chains in corresponding posets.

By means of that equivalence it has been proven in [2] (se also [1]) that chain minor is a wqo of finite posets. The proof uses substantially the fact that any 'poset language' $\mathscr{L}(P)$ consists of strings without repetitions. The problem whether $\preccurlyeq$ is a wqo for languages in general was posed [2]. Generalizing the approach in [2] we answer this question affirmatively.

Theorem 1.2. *$(S^{**}, \preccurlyeq)$ is a wqo.*

One can define a stronger quasiordering $\preccurlyeq^*$ if the mapping f in the definition of $\preccurlyeq$ is injective in addition. We prove that $\preccurlyeq^*$ is a wqo as well.

Theorem 1.3. *$(S^{**}, \preccurlyeq^*)$ is a wqo.*

In Section 2 we give some preliminaries and demonstrate in a simple case our method. Theorems 1.2 and 1.3 are proved in Sections 3 and 4, respectively. In Section 5 we give counterexamples showing that requiring an injective φ in Definition 1.1 destroys the wqo property.

2. Absolute minimum about wqo

Any transitive and reflexive binary relation is called a *quasiordering* or, shortly, *qo*. If $(Q, \leqslant_Q)$ is a qo then $x <_Q y$ means that $x \leqslant_Q y$ and $y \nleqslant_Q x$. A *cone* determined by the element $x \in Q$ is the set $K_x = \{y \in Q \mid y \geqslant_Q x\}$. A qo $(Q, \leqslant_Q)$ is a *well quasiordering* or, shortly, *wqo* if it possesses the property characterized by the following lemma. For the proof and for more background we refer to [4].

Lemma 2.1. *Suppose $(Q, \leqslant_Q)$ is a qo. The following conditions are equivalent.*

(1) *For any infinite sequence $(q_i)_{i=0}^{\infty} \subseteq Q$ there are indices $i < j$ such that $q_i \leqslant_Q q_j$.*

(2) *For any infinite sequence $(q_i)_{i=0}^{\infty} \subseteq Q$ there are indices $0 \leqslant i_0 < i_1 < \cdots$ such that*

$$q_{i_0} \leqslant_Q q_{i_1} \leqslant_Q \cdots .$$

(3) *No infinitely many elements $x_0, x_1, \ldots$ of Q create an antichain or a strictly descending chain*

$$x_0 >_Q x_1 >_Q \cdots .$$

Sequences satisfying (1) are called *good*, other sequences are called *bad*. The infinite monotonic subsequence in (2) is called *perfect*. We recall two folcloric but useful statements.

Cone deleting argument. Suppose $(Q, \leqslant_Q)$ is a wqo and $Q_0, Q_1, \ldots$ are defined by $Q_0 = Q, Q_{i+1} = Q_i \backslash K_{q_i}, q_i \in Q_i$. Then this sequence is finite, $Q_j = \emptyset$ for some j (otherwise $(q_i)_{i=0}^{\infty} \subseteq Q$ would be a bad sequence).

Product argument. Suppose $(Q_i, \leqslant_{Q_i})_{i=0}^{r}$ are wqo's, $Q = Q_0 \times Q_1 \times \cdots Q_r$ and $(Q, \leqslant_{pr})$ is defined by $(x_i)_{i=0}^{r} \leqslant_{pr} (y_i)_{i=0}^{r}$ iff $x_i \leqslant_{Q_i} y_i$ for $i = 0, \ldots, r$. Then $(Q, \leqslant_{pr})$ is a wqo as well (apply Lemma 2.1 $r+1$ times).

Let $(Q, \leqslant_Q)$ be a qo. The *Higman ordering* $(SEQ(Q), \leqslant_H)$ on the set

$$SEQ(Q) = \{(I, \ell) \mid I \text{ is a finite linear ordering and } \ell : I \to Q\}$$

of all finite sequences over Q is defined by $(I_0, \ell_0) \leqslant_H (I_1, \ell_1)$ iff there is an increasing mapping $F: I_0 \to I_1$ such that $\ell_0(x) \leqslant_Q \ell_1(F(x))$ for any $x \in I_0$. We will use the following classical result of the wqo theory [3].

Theorem 2.2 (Higman [3]). *$(SEQ(Q), \leqslant_H)$ is a wqo for any wqo $(Q, \leqslant_Q)$.*

To demonstrate our method in a simple case we prove as an example a weaker version of Higman theorem which deals with the structure $(SET(Q), \leqslant_S)$ consisting of finite subsets of Q with the qo $A \leqslant_S B$ iff there is an injective mapping $F: A \to B$ such that $x \leqslant_Q F(x)$ for any $x \in A$.

Lemma 2.3. *$(SET(Q), \leqslant_S)$ is a wqo for any wqo $(Q, \leqslant_Q)$.*

Proof. We prove by a direct argument that any sequence $A = (A_i)_{i=0}^{\infty} \subseteq SET(Q)$ is good to $\leqslant_S$. We say that $X = (B_i, C_i)_{i=0}^{\infty}$ is a *friend* of A if $(B_i)_{i=0}^{\infty}$ is a subsequence of $A, C_i \subseteq B_i$ for any i, and $(|C_i|)_{i=0}^{\infty}$ is bounded. Set $R(X) = \bigcup_{i=0}^{m} (B_i \backslash C_i)$ and $G(i,x) = |K_x \cap (B_i \backslash C_i)|$ where $x \in Q$. We say that X is a *good friend* of A if in addition $\lim_{i \to \infty} G(i,x) = \infty$ (i.e., for any m there is an n such that $i \geqslant n$ implies $G(i,x) \geqslant m$) for any fixed $x \in R(X)$.

To prove that any A has a good friend we define a (finite) sequence $X_0, X_1, \ldots$ of friends of A and initiate it by $X_0 = (A_i, \emptyset)_{i=0}^{\infty}$. Suppose that $X_k = (B_i, C_i)_{i=0}^{\infty}$ is a friend of A which fails to be a good friend: $G(i_0, x), G(i_1, x), \ldots \leqslant N < \infty$ for some indices $0 \leqslant i_0 \leqslant i_1 < \cdots$ and some $x \in R(X_k)$. Let $D_{i_j} = C_{i_j} \cup (K_x \cap (B_{i_j} \backslash C_{i_j}))$. Then

$$X_{k+1} = (B_{i_j}, D_{i_j})_{j=0}^{\infty}$$

is a friend of A and moreover $R(X_{k+1}) \subseteq R(X_k) \backslash K_x$. According to the cone deleting argument $(X_0, X_1, \ldots)$ terminates in a good friend of A. Notice that when $(|A_i|)_{i=0}^{\infty}$ is bounded then the good friend of A obtained is $(A_{i_j}, A_{i_j})_{j=0}^{\infty}$.

So let $X = (B_i, C_i)_{i=0}^{\infty}$ be a good friend of A. We may assume that $(|C_i|)_{i=0}^{\infty}$ is constant and that $C_0 \leqslant_S C_1 \leqslant_S \cdots$ because by the product argument $(C_i)_{i=0}^{\infty}$ contains a perfect subsequence. Take j sufficiently large such that $G(j,x) \geqslant |B_0 \backslash C_0|$ for any

$x \in B_0 \setminus C_0$. As $C_0 \leqslant_S C_j$ and any $x \in B_0 \setminus C_0$ is majorized (in $\leqslant_Q$) by sufficiently many elements in $B_j \setminus C_j$ we conclude that $B_0 \leqslant_S B_j - A$ is good. □

Recall that A^* is the set of all strings over A and that $\subset$ here is the subsequence relation. The following result is an easy and well-known consequence of Higman theorem.

Corollary 2.4. *Let A be a finite alphabet. Then $(A^*, \subset)$ is a wqo.*

3. Proof of Theorem 1.2

Any finite collection $G = (E, I) = (E(G), I(G)) = (\{e_i \mid i \in I\}, I)$ of finite sets is called a *set system*, elements of E are called *edges*. We permit repetition of edges and for simplicity we omit the indices of edges when possible. If $H = (F, J)$ is another set system such that $F \subseteq E$ (and $J \subseteq I$) then H is said to be a *subsystem* of G. If E consists of mutually disjoint edges then G is said to be a *disjoint system*.

The *matching number* $M(G)$ of $G = (E, I)$ is defined as the maximum number of edges in a disjoint subsystem of G. A *Q-system* is a couple (G, ℓ) where $\ell : E(G) \to Q$ gives to the edges of G labels from the set Q.

Suppose $A = (G_i, \ell_i)_{i=0}^{\infty}$ is a sequence of Q-systems where $(Q, \leqslant_Q)$ is a qo. We say that

$$X = (H_i, \ell_i, H_i')_{i=0}^{\infty}$$

is a *friend* of A if $(H_i, \ell_i)_{i=0}^{\infty}$ is a subsequence of A, H_i' is a subsystem of H_i, and $(M(H_i'))_{i=0}^{\infty}$ is bounded.

We define further

$$R(X) = \bigcup_{i=0}^{\infty} \ell_i(E(H_i) \setminus E(H_i')) \subseteq Q \quad \text{and} \quad G(i, x) = M(H_i''(x)),$$

where $x \in Q$ and $H_i''(x)$ is a subsystem of H_i consisting of the edges

$$\{e \in E(H_i) \setminus E(H_i') \mid \ell_i(e) \in K_x\}.$$

We say that X is a *good friend* of A if in addition

$$\lim_{i \to \infty} G(i, x) = \infty$$

for any $x \in R(X)$.

Lemma 3.1. *Any sequence $A = (G_i, \ell_i)_{i=0}^{\infty}$ of Q-systems labelled by a wqo $(Q, \leqslant_Q)$ has a good friend X.*

Proof. We define again a sequence $X_0, X_1, \ldots$ of friends of A starting with $X_0 = (G_i, \ell_i, \emptyset)_{i=0}^{\infty}$ and show that it terminates in a good friend of A. Suppose $X_k = (H_i, \ell_i, H_i')_{i=0}^{\infty}$ fails to be a good friend of A: $G(i_0, y), G(i_1, y), \ldots \leqslant N < \infty$ for some indices $0 \leqslant i_0 < i_1 < \cdots$ and some $y \in R(X_k)$. Then

$$X_{k+1} = (H_{i_j}, \ell_{i_j}, H_{i_j}' \cup H_{i_j}''(y))_{j=0}^{\infty}$$

is clearly a new friend of A and moreover $R(X_{k+1}) \subseteq R(X_k) \backslash K_y$. According to the cone deleting argument after finitely many steps a good friend of A arises. $\square$

Definition 3.2. A (k,l)-babel where k, l are positive integers is any pair $(\mathscr{B}, A)$ satisfying

1. $\mathscr{B}$ is a babel,
2. $A \subseteq S$, $|A| \leqslant l$,
3. $|S(u) \backslash A| \leqslant k$ whenever $u \in \mathscr{L}$, $\mathscr{L} \in \mathscr{B}$.

Definition 3.3. We denote by S_A^S, $A \subseteq S$, the set of all mappings $\varphi: S \to S$ such that $\varphi | A = id_A$ and $\varphi^{-1}(A) = A$. For two languages $\mathscr{K}$ and $\mathscr{L}$ the notation $\mathscr{K} \preccurlyeq_A \mathscr{L}$ means that $\mathscr{K} \leqslant \varphi(\mathscr{L})$ for some $\varphi \in S_A^S$.

Definition 3.4. Let $\mathscr{L} = (\mathscr{L}_i)_{i=0}^{\infty} \subseteq \mathscr{B}$ be a sequence of languages of a (k,l)-babel $(\mathscr{B}, A)$. Let R be a set of k symbols disjoint to A and let $\chi \in S_A^S$ be fixed such that it maps any $S(u) \backslash A, u \in \mathscr{L}_i, i \geqslant 0$, injectively to R. We introduce the following sequence of Q-systems $P(\mathscr{L}) = (G_i, \ell_i)_{i=0}^{\infty}$ (see "Added in proof", page 88):

$$I(G_i) = \mathscr{L}_i, \quad E(G_i) = \{S(u) \backslash A \,|\, u \in \mathscr{L}_i\}, \quad (Q, \leqslant_Q) = ((R \cup A)^*, \subset),$$

$$\ell_i(e_u) = \chi(u) = \chi(a_0 a_1 \ldots a_m) = \chi(a_0)\chi(a_1) \ldots \chi(a_m).$$

Observation 3.5. To prove Theorem 1.2, it suffices to prove that $((\mathscr{B}, A), \preccurlyeq_A)$ is a wqo for any (k,l)-babel $(\mathscr{B}, A)$.

Proof. If $\mathscr{L} = (\mathscr{L}_i)_{i=0}^{\infty}$ is a sequence of languages and $|S(u)|$, $u \in \mathscr{L}_i$, $i \geqslant 0$, is not universally bounded then $|S(u_0)|$, for some u_0 in some $\mathscr{L}_i$, is at least as big as the sum of lengths of the strings in $\mathscr{L}_0$. Then it is easy to embed the whole $\mathscr{L}_0$ in this single string u_0 and $\mathscr{L}$ is good. Otherwise $|S(u)| \leqslant c$ for all $u \in \mathscr{L}_i$ and all $i \geqslant 0$ and hence $\mathscr{L}$ is a $(c, 0)$-babel. $\square$

Lemma 3.6. *$((\mathscr{B}, A), \preccurlyeq_A)$ is a wqo for any (k,l)-babel $(\mathscr{B}, A)$.*

Proof. We proceed by double induction on k and l and start with $k = 0$. Then $((\mathscr{B}, A), \preccurlyeq_A)$ is a wqo because even $(SET(A^*), \leqslant_S)$ is a wqo by Lemma 2.3 and Corollary 2.4.

Suppose now that $(\mathscr{B}, A)$ is a (k,l)-babel, $k > 0$, and $\mathscr{L} = (\mathscr{L}_i)_{i=0}^{\infty} \subseteq \mathscr{B}$ is a sequence of languages. We prove that $\mathscr{L}$ is good. We may suppose, renaming appropriately symbols, that $S(\mathscr{L}_i)$ are mutually disjoint up to A and that $S \backslash \bigcup_{i \geqslant 0} S(\mathscr{L}_i)$ is infinite.

Let $P(\mathscr{L}) = (G_i, \ell_i)_{i=0}^{\infty}$ be the sequence defined in Definition 3.4. The labels form a wqo by Corollary 2.4. Thus, there is, by Lemma 3.1, a good friend $(H_i, \ell_i, H_i')_{i=0}^{\infty}$ of $P(\mathscr{L})$.

Let F_i be a maximum disjoint subsystem of H_i' and let $U_i = \bigcup E(F_i)$. Clearly, $|U_i| \leqslant ck$ for some constant c (the bound on matching numbers) for any $i \geqslant 0$. We introduce a set T, $|T| = ck$, of completely new symbols which is disjoint to A and to all $\bigcup E(H_i)$. Let $\rho \in S_A^S$ be such that ρ is an identity on $S \backslash \bigcup_{i \geqslant 0} U_i$ and maps any U_i injectively to T.

Consider now the babel $\mathscr{C} = (\rho(\mathscr{K}_i))_{i=0}^{\infty}$ where $(\mathscr{K}_i)_{i=0}^{\infty}$ is defined by $\mathscr{K}_i = I(H_i')$. We see that, crucially, $(\mathscr{C}, T \cup A)$ is a $(k-1, ck+l)$-babel because any edge of H_i' must intersect U_i. We may suppose, according to the induction hypothesis, that $\rho(\mathscr{K}_0) \preccurlyeq_{A \cup T} \rho(\mathscr{K}_1) \preccurlyeq_{A \cup T} \cdots$.

We compare the first term with the others: there are mappings $\varphi_i \in S_{A \cup T}^S$ and $f_i : \mathscr{K}_0 \to \mathscr{K}_i$, $i \geqslant 1$, such that $\rho(u) \subset \varphi_i(\rho(f_i(u)))$ for any $u \in \mathscr{K}_0$. Let j be such a large number that there are $|E(H_0) \backslash E(H_0')|$ mutually disjoint edges

$$F = \{h_e \mid e \in E(H_0) \backslash E(H_0')\} \subseteq E(H_j) \backslash (H_j')$$

satisfying $\ell_j(h_e) \supset \ell_0(e)$ for any $e \in E(H_0) \backslash E(H_0')$ and moreover any edge of F is disjoint to $S(f_j(\mathscr{K}_0))$.

We take a mapping $\varphi \in S_A^S$ as follows.

- If $x \in S(f_j(\mathscr{K}_0)) \cap U_j$ then $\rho(y) = \rho(x)$ for at most one $y \in U_0$. If it exists we put $\varphi(x) = y$.
- If $x \in S(f_j(\mathscr{K}_0)) \backslash U_j$ then we put $\varphi(x) = \varphi_j(x)$.
- If $x \in h_e$ for $e \in E(H_0) \backslash E(H_0')$ then $\chi(y) = \chi(x)$ for at most one $y \in e$. If it exists we put $\varphi(x) = y$.

Otherwise φ is defined arbitrarily. Clearly, $I(H_0) \leqslant \varphi(I(H_j))$ and we conclude that the sequence $\mathscr{L}$ is good. □

Lemma 3.6 and Observation 3.5 prove Theorem 1.2.

4. Proof of Theorem 1.3

An easy check shows that only in Observation 3.5, we used the fact that the mapping f of the definition of $\preccurlyeq$ had not to be injective. In Lemma 3.6, it has been proven actually that $((\mathscr{B}, A), \preccurlyeq_A^*)$ is a wqo for any (k, l)-babel $(\mathscr{B}, A)$. Now we make the whole proof injective by replacing Observation 3.5 by a finer consideration.

Suppose $\mathscr{L} = (\mathscr{L}_i)_{i=0}^{\infty} \subseteq S^{**}$ is a sequence of languages. We say that $X = (\mathscr{K}_i, \mathscr{K}_i')_{i=0}^{\infty}$ is a *friend* of $\mathscr{L}$ if $(\mathscr{K}_i)_{i=0}^{\infty}$ is a subsequence of $\mathscr{L}$, $\mathscr{K}_i' \subseteq \mathscr{K}_i$, $(|\mathscr{K}_i'|)_{i=0}^{\infty}$ is constant, and $\min\{|S(u)| \mid u \in \mathscr{K}_i'\} \to \infty$ for $i \to \infty$. If moreover $(\max\{|S(u)| \mid u \in \mathscr{K}_i \backslash \mathscr{K}_i'\})_{i=0}^{\infty}$ is bounded then X is said to be a *good friend* of $\mathscr{L}$.

Consider the following property:

(∗) For any c there are in some language $\mathscr{L}_i$ c strings u such that for each of them $|S(u)| \geqslant c$

Lemma 4.1. *Suppose $\mathscr{L} = (\mathscr{L}_i)_{i=0}^{\infty}$ is a sequence of languages not having property (*). Then $\mathscr{L}$ has a good friend.*

Proof. We define then by induction a sequence $X_0, X_1, \ldots$ of friends of $\mathscr{L}$ starting with $X_0 = (\mathscr{L}_i, \emptyset)_{i=0}^{\infty}$. If $X_k = (\mathscr{K}_i, \mathscr{K}'_i)_{i=0}^{\infty}$ fails to be a good friend of $\mathscr{L}$ then $|S(u_{i_j})| \to \infty$ for $j \to \infty$ for some strings $u_{i_j} \in \mathscr{K}_{i_j} \backslash \mathscr{K}'_{i_j}$ and some indices $0 \leqslant i_0 < i_1 < \cdots$. Then

$$X_{k+1} = (\mathscr{K}_{i_j}, \mathscr{K}'_{i_j} \cup \{u_{i_j}\})_{j=0}^{\infty}$$

is a new friend of $\mathscr{L}$. As (*) is violated the growth of $|\mathscr{K}'_i|$ cannot proceed arbitrarily long and after finitely many steps a good friend of $\mathscr{L}$ is obtained. □

Proof of Theorem 1.3. Suppose $\mathscr{L} = (\mathscr{L}_i)_{i=0}^{\infty} \subseteq S^{**}$ is a sequence of languages. If $\mathscr{L}$ has property (*) then $\mathscr{L}_0$ embeds injectively in some $\mathscr{L}_j$. If not then consider a good friend $X = (\mathscr{K}_i, \mathscr{K}'_i)_{i=0}^{\infty}$ of $\mathscr{L}$. The sequence $(\mathscr{K}_i \backslash \mathscr{K}'_i)_{i=0}^{\infty}$ is a $(c, 0)$-babel for some c and by Lemma 3.6 we may suppose it forms a perfect sequence $(\mathscr{K}_0 \backslash \mathscr{K}'_0) \preccurlyeq^* (\mathscr{K}_1 \backslash \mathscr{K}'_1) \preccurlyeq^* \cdots$.

So there are mappings $\varphi_i : S \to S$ and injective mappings $f_i : (\mathscr{K}_0 \backslash \mathscr{K}'_0) \to (\mathscr{K}_i \backslash \mathscr{K}'_i)$, $i \geqslant 1$, such that $u \subset \varphi_i(f_i(u))$ for any $u \in \mathscr{K}_0 \backslash \mathscr{K}'_0$. Now we take such a large j that

$$\min_{u \in \mathscr{K}'_j} |S(u)| \geqslant \sum_{v \in \mathscr{K}_0 \backslash \mathscr{K}'_0} |S(f_j(v))| + \sum_{v \in \mathscr{K}'_0} \operatorname{length}(v).$$

It is easy to extend the injective covering $\mathscr{K}_0 \backslash \mathscr{K}'_0 \preccurlyeq^* \mathscr{K}_j \backslash \mathscr{K}'_j$ to the injective covering $\mathscr{K}_0 \preccurlyeq^* \mathscr{K}_j$. We conclude that $\mathscr{L}$ is good. □

5. Concluding remarks

Now we show that the fact we did not require an injective φ was crucial to obtain wqo. Let $\mathscr{K} \preccurlyeq_* \mathscr{L}$, for two languages $\mathscr{L}$ and $\mathscr{K}$, iff there is an injective $\varphi : S \to S$ such that $\mathscr{K} \leqslant \varphi(\mathscr{L})$. Consider this example.

Example 5.1. The infinite babels

$$\mathscr{B}_0 = \{\{132132\}, \{14213243\}, \{1521324354\}, \{162132435465\}, \ldots\}$$

and

$$\mathscr{B}_1 = \{\{ab, bc, ca\}, \{ab, bc, cd, da\}, \{ab, bc, cd, de, ea\}, \ldots\}$$

are antichains to $\preccurlyeq_*$. Thus, $\preccurlyeq_*$ is not a wqo.

Note that both babels are antichains also in the ordering obtained by replacing in Definition 1.1, $\mathscr{K} \leqslant \varphi(\mathscr{L})$ by $\varphi(\mathscr{K}) \leqslant \mathscr{L}$.

Problem 5.2. Suppose now that a language $\mathscr{L} = u_0u_1 \ldots u_k$ is a finite sequence of strings rather than just a set and put $\mathscr{L} = u_0u_1 \ldots u_k \preccurlyeq \mathscr{K} = v_0v_1 \ldots v_l$ iff there is a mapping $\varphi: S \to S$ and an increasing injection $f: \{0, 1, \ldots, k\} \to \{0, 1, \ldots, l\}$ such that $u_i \subset \varphi(v_{f(i)})$ for all $i = 0, 1, \ldots, k$. Is this $\preccurlyeq$ stil a wqo?

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Added in proof. In Definition 3.4 replace χ by χ_u, we need one injective $\chi_u \in S_A^S$ for each string $u \in \mathscr{L}_i, i \geqslant 0$. One mapping χ clearly cannot in general achieve the goal. This small modification changes nothing in the proof of Lemma 3.6.

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